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Author(s)	Kishimoto, Nobu
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Local Well-posedness Results for 1-D Quadratic Nonlinear Schrödinger Equations

京都大学大学院理学研究科　岸本 展 (Nobu Kishimoto)
Department of Mathematics, Kyoto University

1. Introduction

In this note we study the Cauchy problem for 1-D nonlinear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = N(u) \ , & (x,t) \in \mathbb{R} \times [-T,T] \ , \\ u(x,0) = u_0(x) \ , \end{cases} \tag{1}$$

where the solution u is complex valued and the initial data u_0 are given in a Sobolev space $H^s(\mathbb{R})$. We consider a quadratic nonlinearity $N(u) = \overline{u}^2$. Our interest is to establish the time-local well-posedness of (1) under low regularity of data.

The Cauthy problem with low-regularity data for various nonlinear evolution equations has been extensively studied in recent years. It is presumed that the nonlinear effect becomes larger if the Cauchy problem is considered under low regularity, so we can expect that a deep analysis of the nonlinear interactions will be done through the studies of the Cauchy problem in weak spaces.

For the cases of 1-D and the quadratic nonlinearity, the Fourier restriction norm method introduced by Bourgain [2] is considered as a powerful machine to establish the well-posedness under negative Sobolev regularities. Bourgain space $X^{s,b} = X^{s,b}_{x,t}(\mathbb{R}^2)$ is defined for $s, b \in \mathbb{R}$ as the completion of smooth functions on $\mathbb{R}^2$ via the norm

$$\|u\|_{X^{s,b}} := \left\| \langle\xi\rangle^s \langle\tau - \xi^2\rangle^b \, \widehat{u} \right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)},$$

where $\langle\xi\rangle := (1+\xi^2)^{1/2}$ and $\widehat{\cdot}$ denotes the spacetime Fourier transform. By an iteration argument in $X^{s,b}$, Kenig, Ponce and Vega [3] obtained the local well-posedness of (1) with the quadratic nonlinearities $N(u) = u^2$, $\overline{u}^2$ and $u\overline{u}$ for the Sobolev regularity $s > -3/4$, $s > -3/4$ and $s > -1/4$, respectively. In the proof of these results, the key fact is the following bilinear estimate corresponding to the nonlinearity.

Theorem 1. (Kenig-Ponce-Vega [3]) *Consider one of the following cases, (i) $N(u) = u^2$ and $s \in (-3/4, 0]$, or (ii) $N(u) = \overline{u}^2$ and $s \in (-3/4, 0]$, or (iii) $N(u) = u\overline{u}$ and $s \in (-1/4, 0]$. Then there exists $b > 1/2$ such that*

$$\|B(u,v)\|_{X^{s,b-1}} \le C \, \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}, \tag{2}$$

where

$$B(u,v) := \begin{cases} uv & \text{if } N(u) = u^2, \\ \overline{uv} & \text{if } N(u) = \overline{u}^2, \\ u\overline{v} & \text{if } N(u) = u\overline{u}. \end{cases}$$

However, (2) is known to fail for s less than or equal to these thresholds [3,5]. Note that in the frequency space the bilinear estimate (2) is rewrited as

$$\left\| \langle\tau - \xi^2\rangle^{-1} \mathcal{B}(f,g) \right\|_{\hat{X}^{s,b}} \le C \, \|f\|_{\hat{X}^{s,b}} \|g\|_{\hat{X}^{s,b}}, \tag{3}$$

$$\mathcal{B}(f,g) = \begin{cases} f * g & \text{if } N(u) = u^2, \\ \widetilde{f} * \widetilde{g} & \text{if } N(u) = \overline{u}^2, \\ f * \widetilde{g} & \text{if } N(u) = u\overline{u}, \end{cases}$$

where $\widetilde{f}$ is the reflection of f ; $\widetilde{f}(\xi,\tau) := f(-\xi,-\tau)$, $f * g$ denotes the spacetime convolution and $\hat{X}^{s,b}$ is a Banach space with norm

$$\|f\|_{\hat{X}^{s,b}} := \left\| \langle\xi\rangle^s \langle\tau - \xi^2\rangle^b f \right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.$$

In order to lower the regularity bound for well-posedness, we need to modify the space $\hat{X}^{s,b}$ appearing in (3).

In the case of u^2, Bejenaru and Tao [1] subtly modified $\hat{X}^{s,b}$ according to the special property of the nonlinearity and obtained a further local well-posedness result.

Theorem 2. (Bejenaru-Tao [1]) *The Cauchy problem* (1) *with* $N(u) = u^2$ *is locally well-posed in* $H^s(\mathbb{R})$ *for* $s \geq -1$.

They added to the norm a weight factor adapted to the nonlinearity u^2, which is the key to this improvement. Of course their space will not work when we treat other nonlinearities, but it turns out that a suitably weighted norm is still useful when we consider the nonlinearity $\overline{u}^2$. Our main result is then as follows.

Main Theorem. *The Cauchy problem* (1) *with* $N(u) = \overline{u}^2$ *is locally well-posed in* $H^s(\mathbb{R})$ *for* $s \geq -1$.*1

This note is organized as follows. The result on u^2 by [1] is recalled in Section 2. We give an typical example of nonlinear interaction difficult to estimate in $\hat{X}^{s,b}$, and then see how to handle it with a weighted norm. In Section 3, we find out a weight suitable for the case of $\overline{u}^2$, instead of proving Main Theorem or the key bilinear estimate. The details of the proof are provided in [4].

2. Bejenaru and Tao's result ($N(u) = u^2$)

In the case of nonlinearities u^2 or $\overline{u}^2$, the following pair of f, g is one of the typical interactions which cause trouble to the bilinear estimate (3) below $s = -3/4$ (see Figure 1).

*1 In the case of u^2, Bejenaru and Tao also established an ill-posedness result below $s = -1$, so their well-posedness bound $s \geq -1$ is sharp. This sharpness also holds in our case of $\overline{u}^2$ by the same argument.

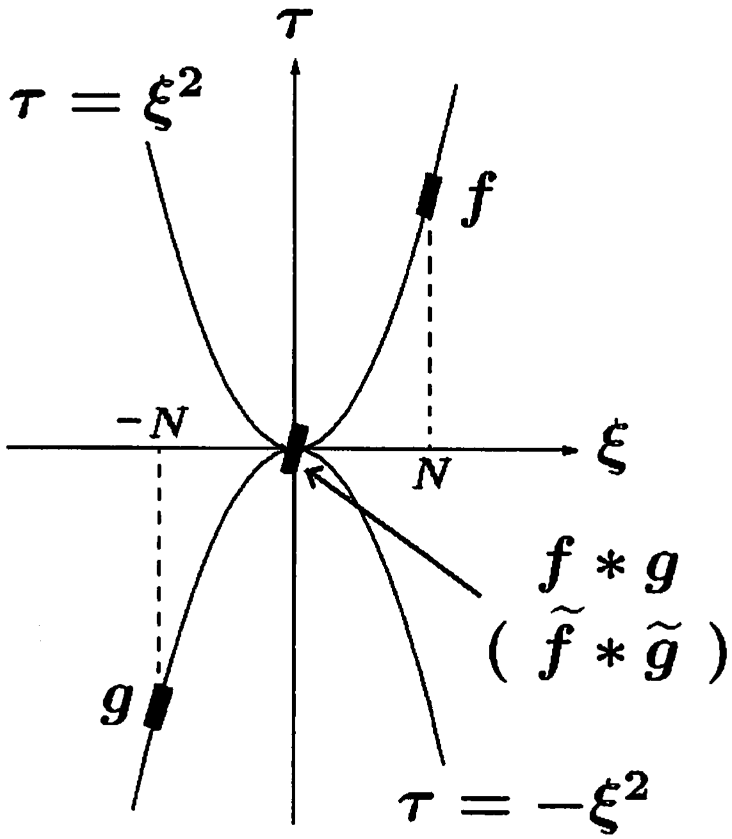

Figure 1

In this example, f and g are supported near the parabola $\{\tau = \xi^2\}$ and the reflected parabola $\{\tau = -\xi^2\}$, respectively, and the output ($f * g$ or $\widetilde{f} * \widetilde{g}$) is near the origin. As N tends to ∞, this is actually a counterexample to (3) with s less than $-3/4$ and some b.

As mentioned above, Bejenaru and Tao handled this interaction by introducing some weight factor to the norm. To see this, we begin with a heuristic observation. Assume that $u(x,t)$ solves the following integral equation associated to (1),

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')[u^2(t')]\,dt',$$

where $\{U(t)\}_{t\in\mathbb{R}}$ is the free Schrödinger group. We formally calculate the Fourier transform of this equation (ignoring the issue of extending u globally in time) to have the formal equation of $\widehat{u}$,

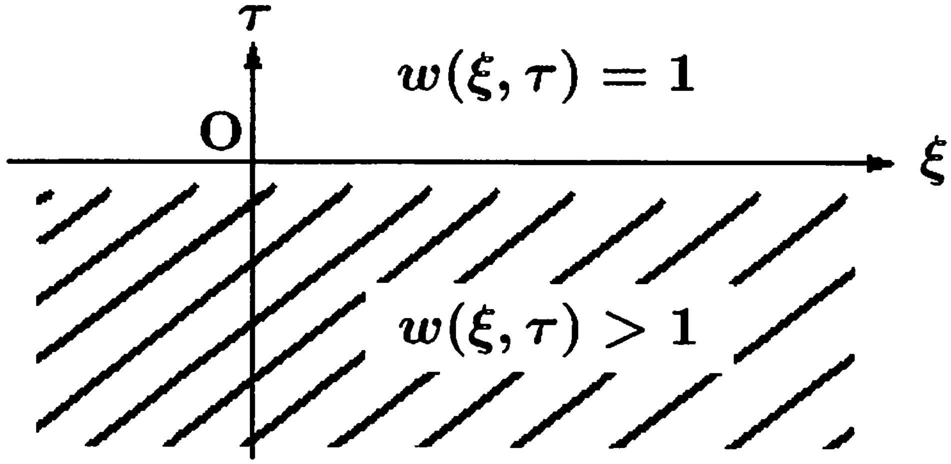

Figure 2

$$\widehat{u}(\xi,\tau) = \delta(\tau-\xi^2)\left\{\widehat{u_0}(\xi) + c\int_{\mathbb{R}} \frac{(\widehat{u}*\widehat{u})(\xi,\tau')}{\tau'-\xi^2}\,d\tau'\right\} + \frac{1}{\tau-\xi^2}(\widehat{u}*\widehat{u})(\xi,\tau), \tag{4}$$

where δ is the Dirac delta function. Let us try to solve (4) iteratively, starting from $\widehat{u}^{(0)} = 0$, then we see that all the iterates $\widehat{u}^{(j)}$ are restricted to the upper half-plane $\{\tau \geq 0\}$. Thus, we expect that the Fourier transform of a solution to (1) is also concentrated on the upper half-plane.

According to this observation, they introduced a new norm $\|f\|' := \|wf\|_{\hat{X}^{s,b}}$, where the weight $w(\xi,\tau)$ is equal to 1 on the upper half-plane and takes large value on the lower half-plane (see Figure 2).*2 The previous dangerous interaction can be controlled by this norm because the norm of g, which is supported on the lower half-plane, becomes larger and increases the right hand side of the bilinear estimate. Since the solution is expected to remain on the upper half-plane, the new

*2 Exactly, they used $w = 1$ $(\tau \geq -1)$, $|\tau|^{10}$ $(\tau < -1)$.

function space is still appropriate for seeking the solution.

With this weighted space (and some more modifications), they established the bilinear estimate and the local well-posedness in H^{-1}.

3. Main result ($N(u) = \overline{u}^2$)

Consider the case of nonlinearity $\overline{u}^2$ next. We again take the Fourier transform of the integral equation

$$u(t) = U(t)u_0 - i \int_0^t U(t-t')[\overline{u^2(t')}]\, dt'$$

to have (at least formally)

$$\begin{aligned}\widehat{u}(\xi,\tau) = \delta(\tau-\xi^2)\left\{\widehat{u_0}(\xi) + c\int_{\mathbb{R}} \overline{\frac{(\widetilde{\widehat{u}} * \widetilde{\widehat{u}})(\xi,\tau')}{\tau'-\xi^2}\, d\tau'}\right\} \\ + \frac{1}{\tau-\xi^2}\overline{(\widetilde{\widehat{u}} * \widetilde{\widehat{u}})(\xi,\tau)},\end{aligned} \tag{5}$$

and there are $\widetilde{\widehat{u}}$'s. If $\widehat{u}$ lay in the upper half-plane, then the first and the second terms of the right hand side would lie on the upper and lower half-plane, respectively, which is a contradiction. Thus we can no longer expect the solution to be supported on the upper half plane.

We want to use some weighted norm again, then what part of the plane is the solution mainly supported in? We again rely on a heuristic observation. Start from zero solution and solve (5). The first iterate $\widehat{u}^{(1)}$ lies on the parabola $\{\tau = \xi^2\}$. Then, how about the second iterate $\widehat{u}^{(2)}$? The first term is supported on this parabola, too, but how about the second term $\widetilde{\widehat{u}^{(1)}} * \widetilde{\widehat{u}^{(1)}}$?

It is difficult to consider the convolution of two delta functions, so we assume that $\widehat{u}^{(1)}$ has many small components along the parabola (see Figure 3). Then we see that the support of $\widetilde{\widehat{u}^{(1)}} * \widetilde{\widehat{u}^{(1)}}$ is distributed like Figure 4; for example, the component of $\widehat{u}^{(1)}$ on the origin interacts

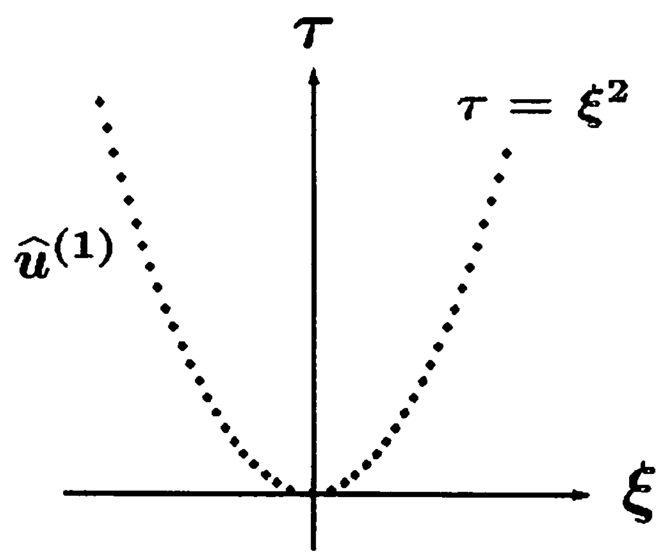

Figure 3

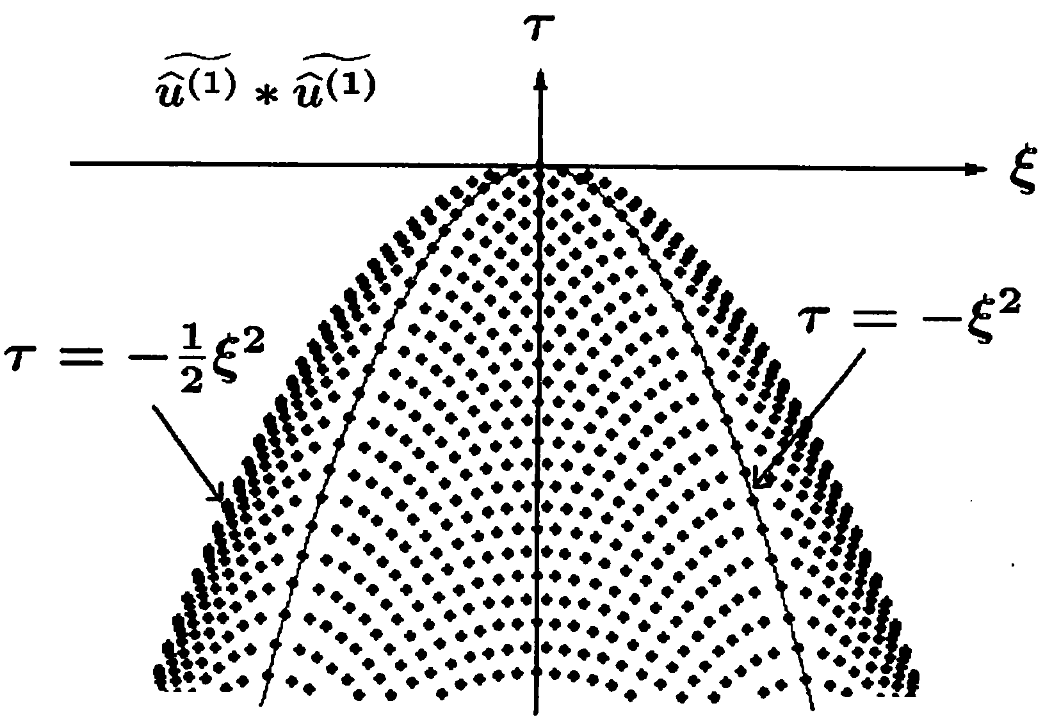

Figure 4

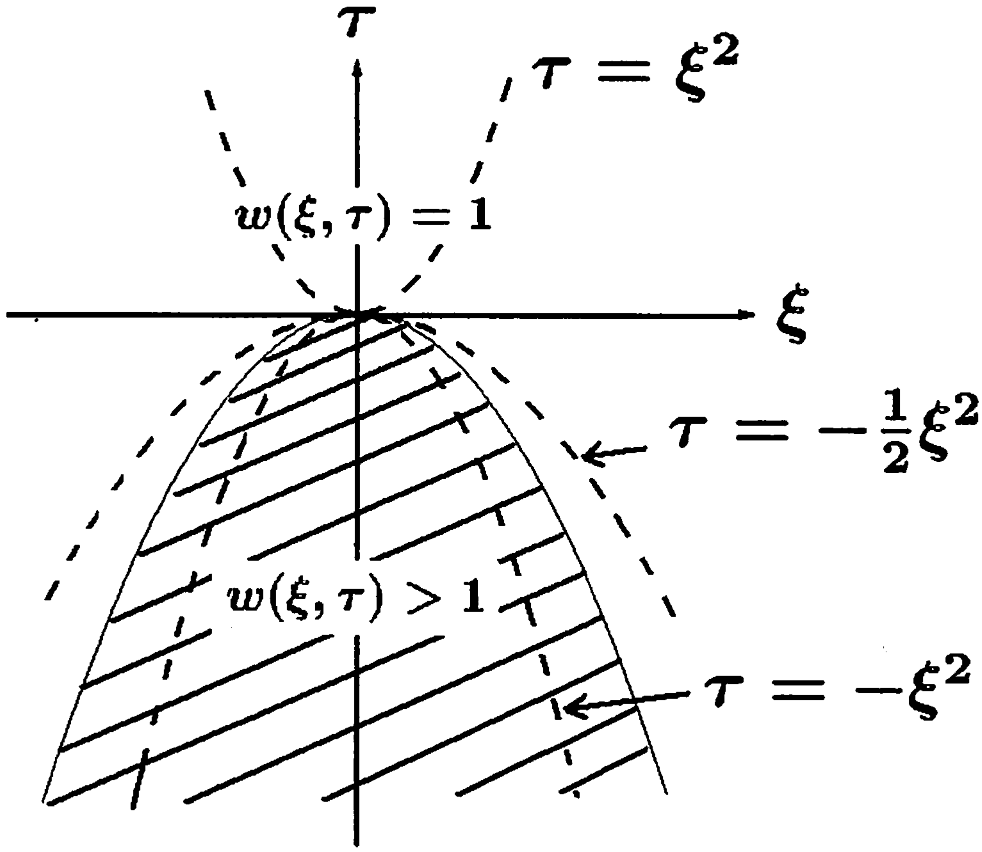

Figure 5

with the other $\widehat{u}^{(1)}$ to make components along the parabola $\{\tau = -\xi^2\}$, and the component on (ξ_0, ξ_0^2) makes components along the parabola $\{\tau + \xi_0^2 = -(\xi + \xi_0)^2\}$.

Carefully observing Figure 4, we notice that $\widetilde{\widehat{u}^{(1)}} * \widetilde{\widehat{u}^{(1)}}$ has high density near the envelope, which is determined to be another parabola $\{\tau = -\frac{1}{2}\xi^2\}$. As a result, if we consider $\widehat{u}^{(1)} + \widehat{u}^{(2)}$ as an approximate solution, a solution is expected to be mainly supported near the two parabolas, $\{\tau = \xi^2\}$ and $\{\tau = -\frac{1}{2}\xi^2\}$.

Based on this obserbation, we use the weight factor which is equal to 1 near these two parabolas, and takes large value on the region where the previous dangerous interaction occurs, that is, near the parabola $\{\tau = -\xi^2\}$. For instance, the weight on the region $\{\tau < -\frac{3}{4}\xi^2\}$ will

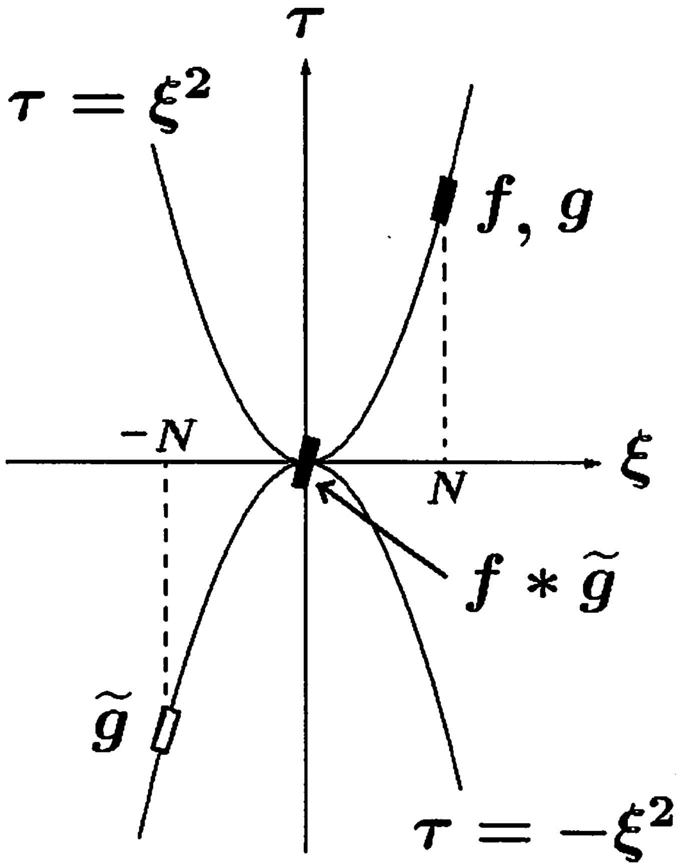

Figure 6

work (see Figure 5).*3

The bilinear estimate in H^{-1} is established by using the function space with the above weight (and some more modifications), and then the local well-posedness result follows. However, in the endpoint case $s = -1$ we need to put some additional weight factor.

Finally, we make a remark on the case of nonlinearity $u\bar{u}$. In this case the typical dangerous interaction occurs between the same high frequency components (see Figure 6). It seems difficult to control this

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*3 Exactly, we use the weight $w = 1$ $(\tau \geq -3\xi^2/4)$, $\langle \tau - \xi^2 \rangle^{1/4}$ $(\tau < -3\xi^2/4)$. We cannot take the weight of the power more than $1/4$ in the region near $\{\tau = -\xi^2\}$, because of another dangerous interaction that does not occur in the case of nonlinearity u^2.

interaction by using some weighted norm, because the region of this interaction coinsides with the region near the parabola $\{\tau = \xi^2\}$, where the solution is expected to have the highest density.

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E-mail address : n-kishi@math.kyoto-u.ac.jp